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# Geometry and the Pure Visual Field

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The thesis which I wish to present and defend in this paper is this: There is a geometry which fits precisely and naturally the configurations of the pure visual field, and that geometry is not a Euclidian geometry but a two-dimensional elliptic, or Riemannian, geometry. I shall present an elaboration and defense of this thesis, then attempt to justify the statement that this geometry "fits naturally" the configurations of the "pure visual field", and finally relate this thesis to previous discussions of the connection between geometries and sensory fields.

## I

By a "pure visual field" I mean a kind of domain of objects which any normal person can be aware of and attend to when his eyes are open - or more exactly, when one of his eyes is open, for I shall be concerned in this paper only with monocular vision. This field is two-dimensional in the following sense: we disregard any consideration of distances between the perceiver and objects of the field. For example, the relations of being "nearer" or "farther away" from the perceiver, of being "in front of" or "behind" another object where this entails one objects' being between the perceiver and another object, or of "bulging towards" or being "indented away from" the perceiver, are not considered. We count as objects of the pure visual field only such two-dimensional colored regions and shapes as a painter might attend to in getting a "good likeness" onto a two-dimensional canvas.

In this pure visual field, then, we distinguish points, lines and regions or areas. But the crucial determinations which make this field subject to geometrical laws are those which measure pure visual distance. By "visual distance" we do not mean the psychologist's "depth perception", which involves assessing a distance between the perceiver and a visual object. Rather we mean the visual width (length, or distance) between two points in the visual field. It is somewhat like the "angular distance" the astronomer measures between two stars. When we think of another person measuring the angular distance, we think of observing that person from above, and noting the angle of the line which travels from object A to the person and then to object B. This angle we identify with the "visual" angle, and its measure is the measure of angular distance. But this way of looking at "visual distance" is misleading; for what _we_ see, when _we_ see a visual distance is not a bent line connecting ourself with each of two objects. What we see is simply a one-directional visual extension (width, length, distance) between two visual objects. These pure visual lengths may be measured rigorously by instruments ( and measured roughly without instruments), and on the basis of such measurements metrical properties may be assigned to visual lines and areas. Fixed units of visual distance are easily established.

Starting with this metrical concept of visual distances, suitably associated with distinct operations and instruments, it is possible to give practically effective and rigorous difinitions of circles and straight lines in the visual field:

> A circle is a closed line such that all points on it are equidistant from a point.
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> A line segment AB is _straight_ if and only if the distance from end-point A to end-point B is equal to the distance from A to any point C on the line, _plus_ the distance from C to B.

From these definitions we may proceed to define angles, and specifically, right angles, then triangles, quadrilaterals (closed figures with four straight lines as sides), rectangles (equiangular quadrilaterals), squares (equilateral rectangles) and so on. Note that the definitions given are all suitable as well for plane or solid Euclidian geometry, as long as the word "distance" is left ambiguous.

Now the question arises whether the geometrical propositions which hold of the objects of the pure visual field belong to Euclidian geometry or not. The answer is plainly that they do not. Two dimensional Euclidian geometry includes such theorems as the following:

1. A straight line cannot be a circle.
2. Every straight line is infinitely extendable.
3. Two straight lines intersect at most in one point.

4. Two straight lines, cut by a third straight line perpendicular to both, never intersect.

5. All equilateral triangles have the same interior angles.

6. The sum of the interior angles of a triangle equals two right angles.

7. The four angles of a rectangle are all right angles.

By precise measurements of visual distance it is shown that none of these theorems hold for the straight lines, triangles and rectangles of the pure visual field. On the contrary,

1'. A straight line can be a circle. (i.e., a visual straight line can be a closed line with all points on it equidistant from a polar point in the visual field).

2'. No straight line is infinitely extendable. (If we extend any straight line segment in the visual field, it eventually returns on itself. It is thus finite, though unbounded).

3'. Every pair of straight lines intersect at two points. (Imagine standing in the middle of a straight railroad track on a vast plain. The visual lines associated with the two rails are demonstrably straight in every segment - they appear perfectly straight, not curved, visually. Yet these visually straight lines meet at two points which are opposite each other on the horizon, and they enclose a substantial region of the visual field.)

4'. Two straight lines, cut by a third straight line perpendicular to both, always intersect. (The two rails, both appearing visually straight, are cut by the straight edge of the railroad tie at our feet and this tie is perpendicular, visually, to both of them; yet the two visual rails intersect twice.)

5'. All equilateral triangles do not have the same interior angles. (Consider large visual triangles, like that between a star due east on the horizon, a star due north on the horizon, and a star directly overhead. In this case equal visual straight lines connect the three stars, so the triangle is equilaterial. Yet the angles are all right angles, and thus are larger than angles of smaller equilateral triangles which approach $60^{\circ}$).

6'. The sum of the interior angles of a triangle are always greater than two right angles. (This was the case in the visual triangle described above, and would be found to be the case for all other visual triangles upon careful measurement).

7'. The four angles of a rectangle are always larger than right angles. (This is clear if we measure the visual angles - that is, the angles appearing in the visual field - of, say, a picture frame if it is visually rectangular. It is not necessary to use instruments to see this. We can approach a picture frame so that the sides are not only all straight, but the angles are all visually equal. Yet the equal angles are all ~~obtuse~~ obtuse, visually.)

Now all of the counter-Euclidian propositions just enumerated are theorems in the two-dimensional non-Euclidian bipolar elliptic geometry of Riemann. In fact, all theorems of that geometry will fit precisely (so far as precision is possible) the configurations of the visual field. We could (but shall not try to do so here) present a rigorous axiomatic formulation. To those familiar with this type of non-Euclidian geometry, the remarks above should be sufficient proof that it describes ~~both~~ rigorously ~~[illegible]~~ the geometry of the pure visual field in monocular vision.

## II

Standard treatments of non-Euclidian geometry usually suggest that these geometries are not incompatible with sense experience by two lines of argument. It is pointed out that we might conceivably find, as we made direct measurements of larger and larger objects, that Euclidian laws began to fail; that, for example, sufficiently large triangles had noticeably more than two right angles as the sum of their interior angles. Or, secondly, we are told of Euclidian models, - hyperbolic saddle-shaped surfaces, or the surface of a sphere - of which, provided we redefine our terms, the non-Euclidian geometries would hold. The first line of argument does not suggest a "natural" or common sense application of non-Euclidian geometry because (with the possible exception of some of the evidence for relativity), no one has found the kind of empirical evidence required. Given straight, rigid bodies as measure standards, and well-calibrated theodolites, the evidence conforms to the view that ~~actual~~ physical triangles in three dimensional space about us have interior angles adding up to two right angles, and belong to a Euclidian geometry. The second suggestion serves well enough as a ~~connecting~~ consistency proof, but it ~~surely~~ scarcely suggests a natural or ordinary application of non-Euclidian geometry since the surface of a sphere belongs to solid Euclidian geometry, and it would be pointless to begin calling arcs of great circles "straight lines", when it seems obvious that they are curved and not straight.

In contrast to these ways of relating elliptical geometry to sense experience, the preceding section suggests a domain of objects which is constantly available to every normal person, where the words "line", "angle", "straight" ~~and~~ "circle", "triangle", "rectangle", etc., have established ordinary uses, where even without instruments we can make reasonably good judgements of comparative (visual) magnitudes, and where the axioms and theorems of two-dimensional bipolar elliptical geometry may be found immediately and incontrovertibly true. Any normal man can agree that visually the two rails of the railroad track truly appeartoto meet in two places, though they also truly appear completely straight. He can distinguish very well whether a line segment appears straight to the vision, or bent, or curved. And he can tell with fair accuracy whether a visual image appears square, rectangular, circular, etc. Our definitions are entirely compatible with the ordinary use of such terms with respect to visual images.

III

That there is the geometry of the pure visual field is a double elliptic geometry seems so obvious once one grasps it, that one feels that this must have been recognized, even taken for granted, by previous thinkers who have spent much time on the relationship between geometry and vision. Yet, if this thesis has been pointed out, I have been unable as yet to find out where.* The science of optics dealt extensively with the notion of light waves travelling in straight lines in three-space from object to eye. It often, but not always, even treated the image found on the retina as a spherical surface; but all of this was not from the point of view of what the perceiver actually perceives in the visual field, but from that of what an outside observer might find to be the configurations of light rays, or those on the eyeball of the perceiver. Again, much was done with perspective. Painters learned to look at objects before them as two-dimensional arrangements of color patches. But the purpose of the painter was to utilize what he thus saw to arrange physical paint on a flat Euclidian plane (his canvas). Thus laws of perspective are formulated in terms of Euclidian geometry. They worked well enough for the painter's purpose to make arrangements on his flat physical canvas - without anyone's needing to note that whoever observed this canvas would actually see its lines and shapes in their visual field, as in conformity with non-Euclidian laws.

One might have expected that Berkeley would have recognized the thesis here

* Since writing this paper, I have found Thomas Reid's chapter on the "Geometry of Visibles" in his Inquiry ~~of the~~ into the Human Mind, which anticipates our account largely.

presented in his Essay Towards a New Theory of Vision. We take his pronouncement early in the essay that "Distance, of itself and immediately, can not be seen" (Cf. §2) as his way of saying that he is dealing with two-dimensional visual areas of colors, without consideration of a third dimension. Yet his essay rejects any significant connection between the field of vision and geometry. He holds that "visible firgures" are "not the objects of geometry", because visible extensions have "no settled determinate greatness" as do tangibly extended objects. To have a geometry, he holds, one must have a common unit of measure. But measurements are always made, he says, by laying tangible extension upon tangible extension (Cf. §151).

Ernst Mach, in Space and Geometry (1906) also passed over the geometry of pure visual space for reasons rather close to those of Berkeley. He had a strong tendency to identify "space" with three-dimensional Euclidean space, in spite of his respect for non-Euclidean geometers. He said,

> Seldom have thinkers become so absorbed in revery, or so far estranged from reality, as to imagine for our space a number of dimensions exceeding the three of the given space of sense, or to coneive of representing that space by any geometry that departs appreciably from the Euclidean. (Space and Geometry, p 135)

And this tendency was closely connected with his arguments that "Geometric concepts are the product of the idealization of physical experiences of space."(p 94). He insists that metrical properties of straight lines depend upon experiences with physical, tangible, objects.

> Measurement is experience involving a physical reaction, and experiment of superimpostion. Visualized or imagined lines having different directions and lengths cannot be applied to one another forthwith. The possibility of of such a procedure must be actually experienced with material objects accounted as unalterable. (p. 62)

Both Mach and Berkeley thus rejected a geometry of the pure visual field on the grounds that there is no way of determining metrical quantities there. But this ground we have shown clearly false. We can construct simple instruments which will provide a fixed standard of the visual widths or distances between points in the pure visual field, e.g., a piece of cardboard on which angles have been marked off around a central point held to the eye. Such instruments, unlike ruler and compass, do not involve physical, or tactile, contact with the objects measured. Yet they provide fixed standards of visual distance, and unlike rulers, even permit units of distance tied to absolute, (i.e., purely geometrical) magnitudes - such as the magnitude a straight visual line makes upon returning upon itself (i.e., $360^\circ$ of visual angle). The fact that such instruments are three-dimensional does not affect the two-dimensionality of the visual field any more than the use of a three-dimensional compass or ruler compromises the two-dimensionality of Euclidean plane geometry. Indeed, theoretically the use of such three-dimensional aids is inessential, they are but props and checks for our too limited sensory retention and acuity. Thus it seems clear that the pure visual field has metrical properties that can be measured precisely, and that both Mach and Berkeley apparently overlooked this.

Mach did indeed see some of the characteristics of the visual field that we have mentioned.

> The space of Euclidean Geometry is everywhere and in all directions constituted alike; it is unbounded and infinite in extent. On the other hand the space of sight, or "visual space", as it has been termed by Johannes Muller and Hering, is found to be neither constituted everywhere and in all directions alike, nor infinite in extent and unbounded.
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> Space and Geometry, p. 5

Thus he recognized that visual space was finite and unbounded, a characteristic of Riemannian space. Still, in speaking of its heterogeneity, Mach was clearly concerned with psychological variations rather than geometrical properties. He mentioned the different "feelings" which are associated with "upness" and "downness", "rightness" and "leftness", and "nearness" and "farness", as facts which showed the heterogeneity of visual space. (E.g., we do not recognize a complex visual form, like a face, when it is upside down). But such considerations are quite irrelevant; granted the ability to measure visual distances we ignore such psychological "feelings" and concentrate on metrical relationships between visual points and areas.

Russell and Nicod might also have been expected to recognize the geometry of the pure visual field. Nicod's Geometry and the Sensible World was very close in spirit to Russell's theory of perspectives, and both writers conceived of constructing an external world from classes of perspectives, or momentary visual fields, such as might be perceived by single individuals. But the geometry of these visual fields was passed over rather lightly by both authors. Nicod indicated, as Mach had, that he recognized certain non-Euclidean aspects of the visual field. But neither he nor Russell explicitly recognized that a complete intuitive exemplification of a non-Euclidean geometry was at hand.

There have been some developments in the last twenty years in psychology in which visual space has been linked with non-Euclidean geometry. Rudolph Luneberg, in his Mathematical Analysis of Binocular Vision (Princeton, 1947) holds that perceived space is non-Euclidean. But his argument is based on certain experiments dealing solely with binocular disparity of images, and pays no attention to the more general properties of visual distance and size in monocular vision that we have considered. Further the type of geometry he ascribes to visual space is hyperbolic, rather than elliptic, and thus apparently ignores completely the kinds of facts we have discussed above. Much closer to an actual statement of our thesis is a passage by Gibson:

> "It is interesting to note that if we could combine all these two dimensional projections of a three dimensional visual world into a single scene, we would obtain a two dimensional space, in the geometrical sense, which is non-Euclidean.
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> "It would have the properties of the theoretical space defined by the surface of a sphere considered as a two-dimensional surface, i.e., it would be boundless and yet finite, and it would return upon itself. ... a point which traces a straight line will eventually come back to the position from which it started..."
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> James J. Gibson, The Perception of the Physical World, 1950, p122.

Yet the fact that Gibson views such a geometry as the result of hypothetical process of "combining all two-dimensional projections ...into a single scene", rather than as a geometry directly applicable to immediate visual objects - as well as the fact that this view is mentioned only briefly on two or three pages in the whole book - suggests that the full significance of his statements are not recognized.

## IV

Among the philosophical problems suggested by the finding that the pure visual field follows the laws of elliptic geometry are the following: Granted that Euclidean Geometry, as a formal system with a standard interpretation, is tied to tactile or physical measurements of length as Berkeley and Mach supposed, if purely visual ~~perception~~ forms conform to non-Euclidean geometry, why or on what grounds, does man believe normally that "real" space is Euclidean? How, or why, given non-Euclidean visual configurations do we translate these into perceptions of objects as three-dimensional and Euclidean? Granted that the standard laws of interposition, parallax or binocular displacement, and so on, provide "clues", why is the geometry of vision subordinated to the geometry of touch? What is suggested about the criterion of "real" or "things" that we are apt to say that real things conform to the laws of Euclid in our ordinary world? Our thesis is not without its implications, I think, either in metaphysics, or epistemology, or in the analysis of language.

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